# Designing a Fair Tontine Annuity

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# Fair Bet

A bet is *fair* if each party's expected gain is 0.

- Flip a coin: heads I win \$1, tails I pay \$1.

$$\text{expected gain } = \frac{1}{2} \cdot 1 - \frac{1}{2} \cdot 1 = 0.$$

- Roll a die: roll a 6 I win \$5, anything else I pay \$1.

$$\text{expected gain } = \frac{1}{6} \cdot 5 - \frac{5}{6} \cdot 1 = 0.$$

# Agenda

- **Tontines**
- Fair Tontine
- Fair Tontine Annuity
- Concluding Remarks

# What's a Tontine?

- At time 0, each of $m$ persons (members) contributes $s$ dollars.
- When a member dies, his $s$ is divided among survivors.
  - After first death, each survivor receives $s/(m-1)$.
  - After second death, each survivor receives $s/(m-2)$.
  - Etc.
- Provides each member a lifetime payment stream.

# What's Wrong with a Tontine?

- *Age discrimination.* Older members have lower expected payout.
- *Identical contribution level.* Too low for the rich, too high for the middle class.
- *Closed end.* All members join at time 0, then nobody else joins.
- *Increasing payments.* Starve during early years.

# What We Are Going To Do

- Make the tontine fair for all ages and contributions.
- Wrap the tontine with features that make it resemble an annuity.
- Result: a Fair Tontine Annuity.

# Remark

- For simplicity, will mostly ignore the time value of money.
  - Will assume interest rate of 0%.

- Will discuss it at end of talk.

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# Tontine with Unequal Payouts

- Say we have $m$ members, indexed $1, 2, \ldots, m$.
  (Different ages, genders, contributions.)

- Member $i$ contributes $s_i$ dollars.

- If member $j$ dies, member $i$ receives $\alpha_{ij} s_j$.

  - $0 \leq \alpha_{ij} \leq 1$ for $i \neq j$. Member $i$ receives some fraction of $j$'s contribution.

  - $\alpha_{jj} = -1$. Member $j$ forfeits entire contribution.

  - $\sum_{i=1}^{m} \alpha_{ij} = 0$. Amounted forfeited by $j$ equals amount received by surviving members.

# Matrix Representation of $\alpha_{ij}$'s

For example, $m = 5$:

$$\begin{matrix} -1 & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} & \alpha_{1,5} \\ \alpha_{2,1} & -1 & \alpha_{2,3} & \alpha_{2,4} & \alpha_{2,5} \\ \alpha_{3,1} & \alpha_{3,2} & -1 & \alpha_{3,4} & \alpha_{3,5} \\ \alpha_{4,1} & \alpha_{4,2} & \alpha_{4,3} & -1 & \alpha_{4,5} \\ \alpha_{5,1} & \alpha_{5,2} & \alpha_{5,3} & \alpha_{5,4} & -1 \end{matrix}$$

- If member $j$ dies, column $j$ describes payouts.
  - Column elements sum to 0.

- Call the $\alpha_{ij}$'s a *transfer plan*.

# Making the Transfer Plan Fair

- Suppose at time $t$ a member dies.
  - Pretend we don't know which member has died. Let

    $$p_j = \text{Pr}\{\text{member } j \text{ died at } t \,|\, \text{some member died at } t\}$$

  - Remark: $p_j = \mu_j(t)/\sum_k \mu_k(t)$, where $\mu_j(t)$ is force-of-mortality for $j$
- Expected amount received by member $i$ is

  $$ER_i = \sum_{j=1}^{m} p_j \alpha_{ij} s_j.$$

- Plan is fair if $ER_i = 0$ for each member $i$.
  - We call any such plan a *Fair Transfer Plan (FTP)*.

# Recap of FTP Conditions

$$\alpha_{jj} = -1 \text{ for } j = 1, 2, \ldots, m;$$

$$0 \leq \alpha_{ij} \leq 1 \text{ for } i, j = 1, 2, \ldots, m,\ i \neq j;$$

$$\sum_{i=1}^{m} \alpha_{ij} = 0 \text{ for } j = 1, 2, \ldots, m;$$

$$\sum_{j=1}^{m} \alpha_{ij} p_j s_j = 0 \text{ for } i = 1, 2, \ldots, m.$$

# Does an FTP Exist?

- A necessary condition for FTP existence is:

$$0 = \sum_{j=1}^{m} \alpha_{ij} p_j s_j = -p_i s_i + \sum_{j \neq i} \alpha_{ij} p_j s_j \leq -p_i s_i + \sum_{j \neq i} p_j s_j,$$

so

$$2 p_i s_i \leq \sum_{j=1}^{m} p_j s_j \text{ for } i = 1, 2, \ldots, m.$$

- The condition is also sufficient, but the proof is harder and omitted here.
  - Can be shown using network flow graph and max-flow min-cut theorem.

# FTP Existence Theorem

**Theorem.** *An FTP exists if and only if*

$$p_i s_i \leq \frac{1}{2} \sum_{j=1}^{m} p_j s_j \text{ for } i = 1, 2, \ldots, m.$$

- $p_i s_i$ is member $i$'s risk of loss.
  - Elderly (high $p_i$) with large contribution (high $s_i$) at high risk of loss. Younger with small contribution at low risk.
- Theorem says risk of loss cannot be concentrated in any one member.
- Easy to meet in practice by capping contribution.

# How to Construct an FTP

- Many solutions to FTP constraints, some better than others.

- A good solution: "separable" FTP.

  – Assign each member $i$ a weight $w_i \geq 0$, $\sum_{i=1}^{m} w_i = 1$.

  – Distribute $s_j$ in proportion to $w_i$:

$$\alpha_{ij} = \frac{w_i}{(1 - w_j)} \text{ for } i \neq j.$$

  – Such an FTP exists, is unique, and has good properties.

  – The $w_i$'s can be computed fast, $O(m)$ run time.

# How to Make a Fair Tontine

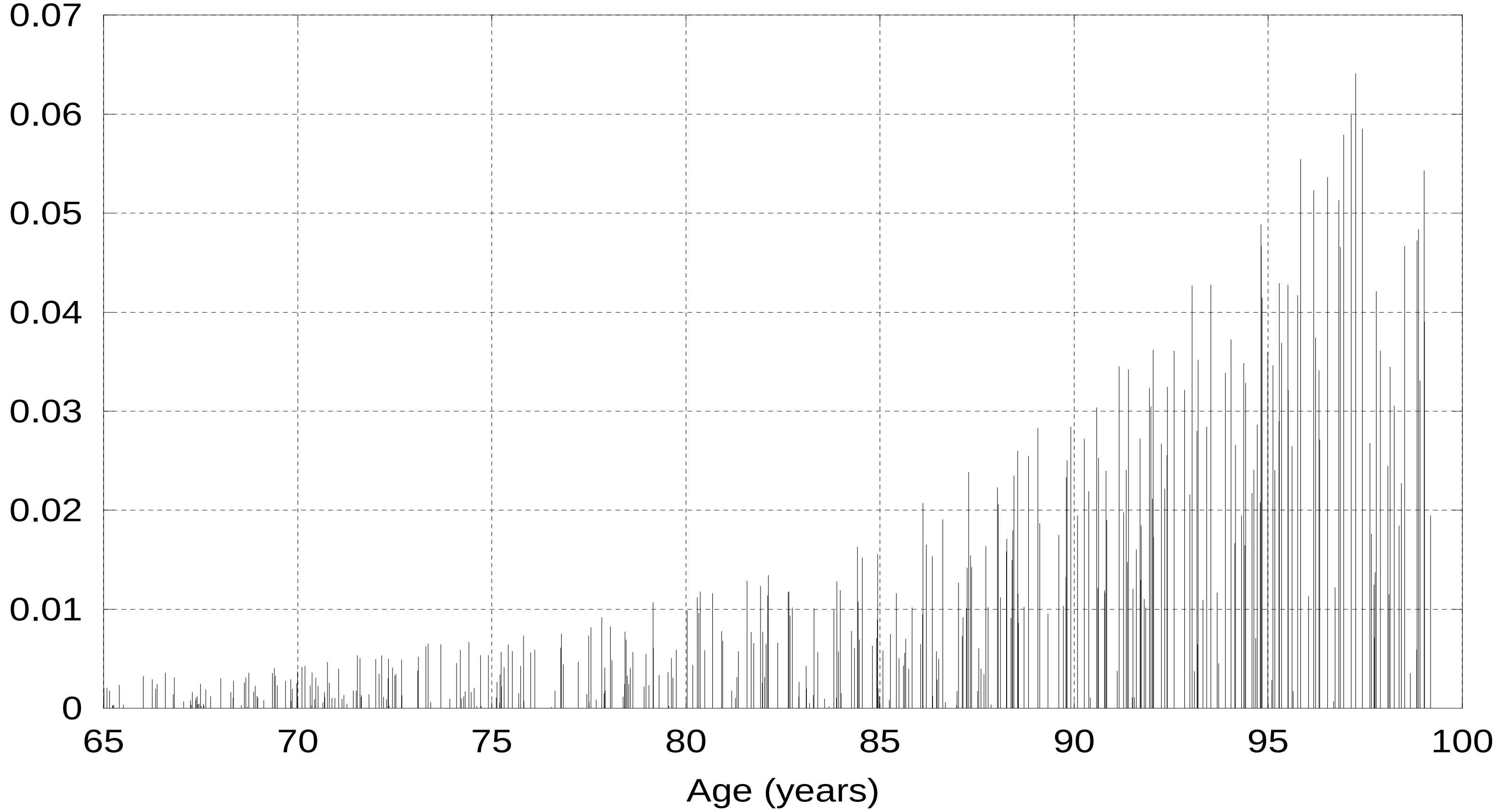
When a member dies:

- Count $m$, compute $p_j$'s, construct an FTP.
- Distribute dying member's contribution according to FTP.

Accomplishes the following goals:

- *Age indiscriminate.* All ages treated fairly.
- *Individual contributions.* All contribution levels treated fairly.
- *Open ended.* New members may join at any time. Tontine operates in perpetuity.

# Fair Tontine Simulation




Payments received by a typical long-lived male, normalized to $1 contribution.

About 5,200 members, wide range of ages, genders, and contributions.

# How to Get Monthly Payments in a Tontine

- Each member has an account that holds her contribution.

- When a member dies, his account distributed using an FTP.
  - Distribution deposited into accounts of surviving members.
  - The $s_i$ values for FTP are account balances, including any deposits from prior FTP's.

- At end of month, each living member is paid balance of her account in excess of her contribution.

# Example Statement of Tontine Account: Living Member

| Date | Amount | Balance | Description |
|---|---:|---:|---|
| 03/31 | | 250,000.00 | |
| 04/02 | 67.17 | 250,067.17 | Proceeds from FTP |
| 04/03 | 25.21 | 250,092.38 | Proceeds from FTP |
| 04/05 | 55.14 | 250,147.52 | Proceeds from FTP |
| 04/07 | 135.41 | 250,282.93 | Proceeds from FTP |
| 04/07 | 48.91 | 250,331.84 | Proceeds from FTP |
| 04/12 | 52.29 | 250,384.13 | Proceeds from FTP |
| 04/15 | 102.54 | 250,486.67 | Proceeds from FTP |
| 04/20 | 159.46 | 250,646.13 | Proceeds from FTP |
| 04/21 | 139.68 | 250,785.82 | Proceeds from FTP |
| 04/22 | 17.82 | 250,803.63 | Proceeds from FTP |
| 04/25 | 124.81 | 250,928.44 | Proceeds from FTP |
| 04/28 | 55.32 | 250,983.76 | Proceeds from FTP |
| 04/30 | 57.91 | 251,041.67 | Proceeds from FTP |
| 04/30 | (1,041.67) | 250,000.00 | Payout of FTP proceeds |

# Example Statement of Tontine Account: Deceased Member

| Date | Amount | Balance | Description |
|---|---|---|---|
| 03/31 | | 250,000.00 | |
| 04/02 | 67.17 | 250,067.17 | Proceeds from FTP |
| 04/03 | 25.21 | 250,092.38 | Proceeds from FTP |
| 04/05 | 55.14 | 250,147.52 | Proceeds from FTP |
| 04/07 | 135.41 | 250,282.93 | Proceeds from FTP |
| 04/07 | 48.91 | 250,331.84 | Proceeds from FTP |
| 04/12 | (250,331.84) | 0 | Forfeited to FTP |

# Expected Value of Monthly Payment

Let:

$$q = \text{Pr}\{\text{die during month} \mid \text{alive at start of month}\}$$
$$r = E(\text{payment} \mid \text{alive at end of month})$$

If member dies during month, loses $s$;
if survives month, gains payment. By fairness:

$$-sq + r(1-q) = 0,$$

so

$$r = \frac{sq}{1-q}.$$

Surprise! Expected payment depends only on member's own age and gender (for $q$) and contribution $s$. Parameters of other members do not matter.

# Fair Tontine Theorem

**Theorem.** *In a fair tontine with monthly payments and original contribution* $s$*, let*

$$q = \text{Pr}\{\textit{die during month} \,|\, \textit{alive at start of month}\}.$$

*Then the expected payment, given that the member survives the month, is*

$$\frac{sq}{1-q}.$$

# Agenda

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# How to Get Constant Expected Payments: "Self Payback"

- Each month, reduce a living member's account balance by paying him a portion of initial contribution.
  - In addition to paying out FTP proceeds for the month.

- Combination of this (deterministic) self payback plus the (random) FTP proceeds has an expected value that is constant over member's lifetime.

- This arrangement is called a *Fair Tontine Annuity* (FTA).

# Example Statement of FTA Account: Living Member

| Date | Amount | Balance | Description |
|---|---|---|---|
| 03/31 | | 250,000.00 | |
| 04/02 | 67.17 | 250,067.17 | Proceeds from FTP |
| 04/03 | 25.21 | 250,092.38 | Proceeds from FTP |
| 04/05 | 55.14 | 250,147.52 | Proceeds from FTP |
| 04/07 | 135.41 | 250,282.93 | Proceeds from FTP |
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| 04/28 | 55.32 | 250,983.76 | Proceeds from FTP |
| 04/30 | 57.91 | 251,041.67 | Proceeds from FTP |
| 04/30 | (1,041.67) | 250,000.00 | Payout of FTP proceeds |
| 04/30 | (452.18) | 249,547.82 | Self payback |

# How Much Self Payback in an FTA?

Define month $n$ as $(nT, (n+1)T)$, $T = 1/12$.
Let:

$$\begin{aligned} D_n &= \text{payment at end of month } n \\ &\quad \text{(self payback plus FTP proceeds);} \\ c &= \text{desired value of } E(D_n \,|\, \text{alive at end of month } n); \\ b_n &= \text{account balance at start of month } n \\ &\quad \text{(self payback at end of month } n \text{ is } b_n - b_{n+1}). \end{aligned}$$

Claim: correct amount of self payback occurs when

$$b_n = \sum_{k=n+1}^{\infty} c \,\Pr\{\text{alive at } kT \,|\, \text{alive at } nT\}.$$

# How Much Self Payback, cont'd

Proof of claim. Let $p(k|n) = \Pr\{\text{alive at } kT \,|\, \text{alive at } nT\}$. Then

$$b_n = \sum_{k=n+1}^{\infty} c\,p(k|n),$$

and

$$\begin{aligned} & E(D_n \,|\, \text{alive at end of month } n) \\ &= \underbrace{\frac{b_n(1 - p(n+1|n))}{p(n+1|n)}}_{\text{from FTPs}} + \underbrace{b_n - b_{n+1}}_{\text{self payback}} \\ &= \frac{b_n}{p(n+1|n)} - b_{n+1} \\ &= \sum_{k=n+1}^{\infty} c\,p(k|n+1) - \sum_{k=n+2}^{\infty} c\,p(k|n+1) \\ &= c. \end{aligned}$$

# FTA Theorem

Notice that the starting balance is:

$$b_0 = \sum_{k=1}^{\infty} c \, \mathrm{Pr}\{\text{alive at } kT \,|\, \text{alive at } 0\}.$$

Surprise! $b_0$ is identical to premium of a fair annuity having monthly payment $c$.

Thus we have established:

**Theorem.** *An FTA with initial contribution $s$ has a random monthly payment whose expected value is identical to the fixed monthly payment of a fair annuity purchased for premium $s$.*

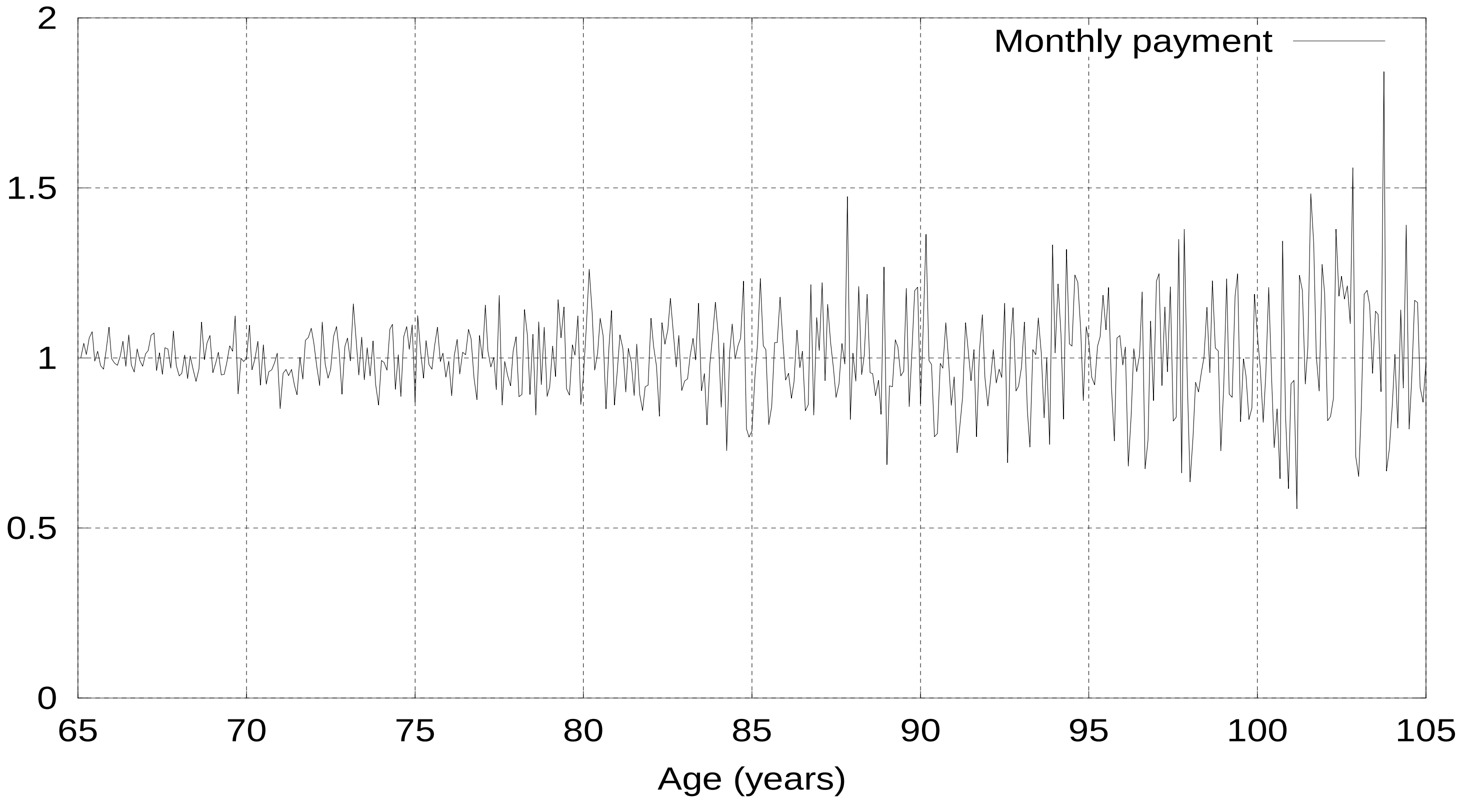
# Remarks on FTA Theorem

- FTA is a noisy version of a fair annuity:
  expected monthly payment of FTA
  $=$ monthly payment of fair annuity.

- Expected monthly payment of FTA
  $>$ monthly payment of insurer-provided annuity

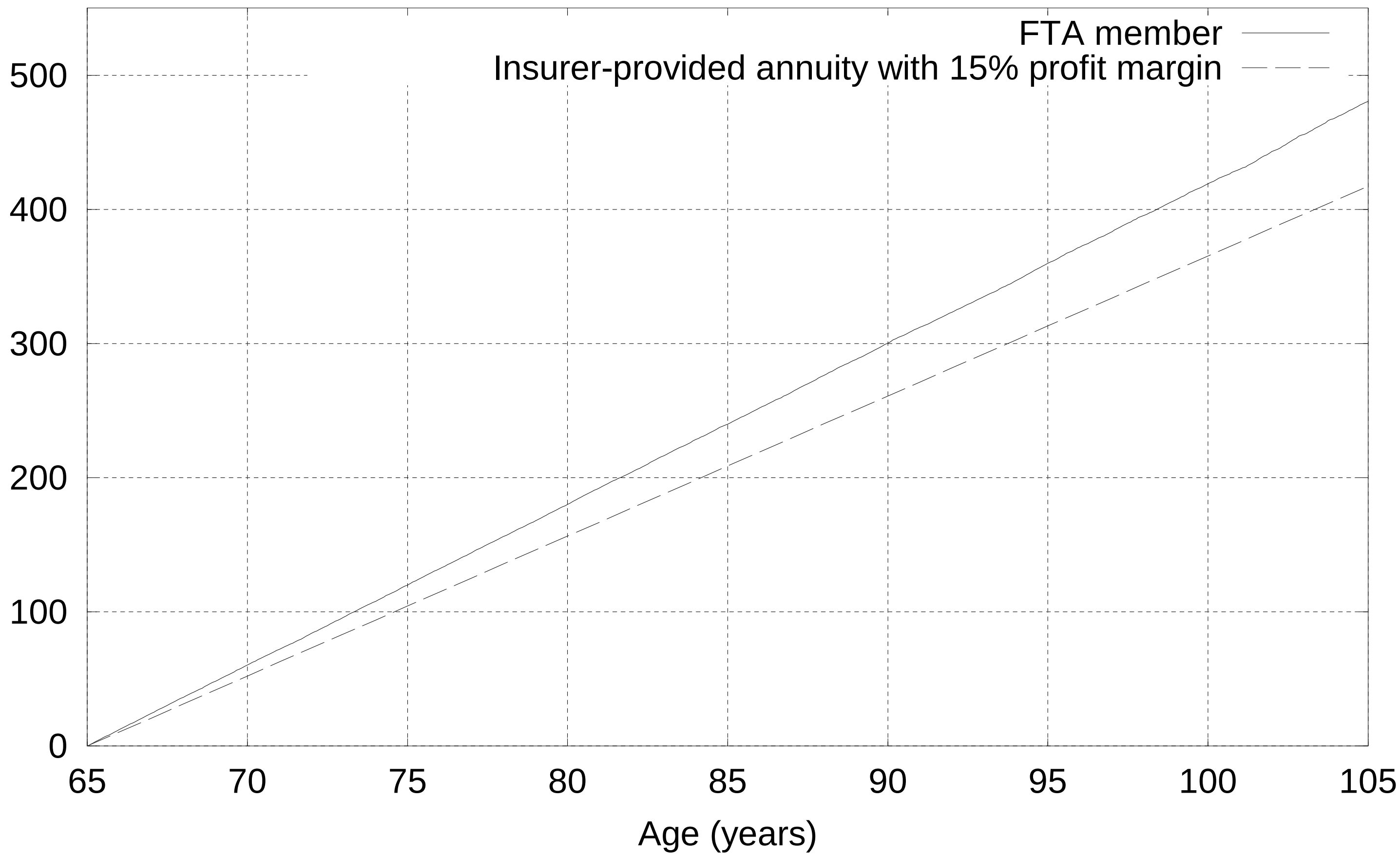
- Law of large numbers, for large pool of members:
  monthly payment of FTA
  $\approx$ monthly payment of fair annuity.

# FTA Simulation



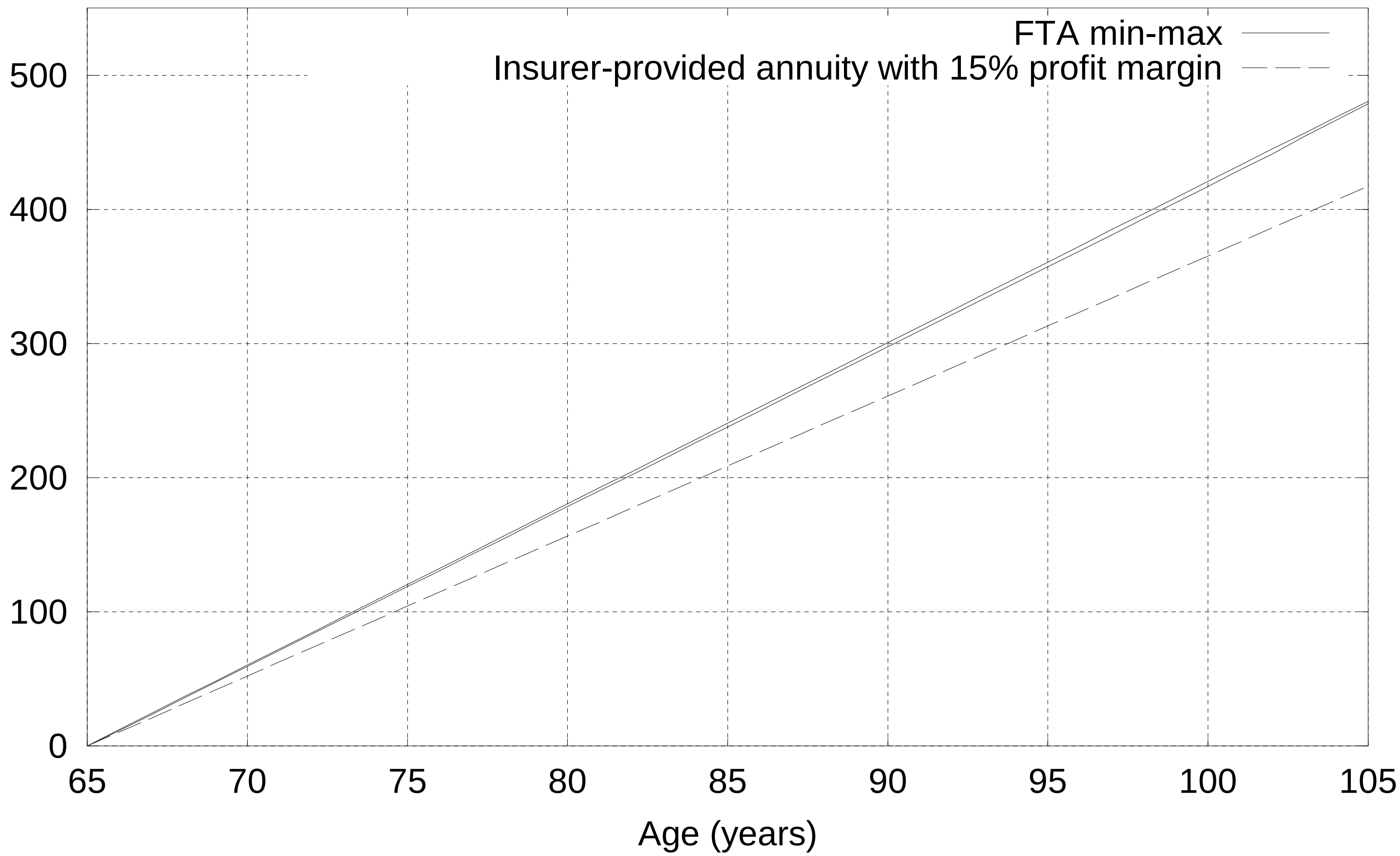

Monthly payment for a typical long-lived member, normalized to $1 expected value, versus age.
About 5,200 members, wide range of ages, genders, and contributions.

# FTA Simulation




Accumulated normalized payout versus age: a typical long-lived FTA member; insurer annuity with typical profit margin.

# FTA Simulation




Accumulated normalized payout versus age: max-min over all FTA members; insurer annuity with typical profit margin.

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# Recap

- A tontine can be made fair using FTPs.

- FTP exists if no one member's assets dominate.

  - Easily constructed (e.g., separable algorithm).

- In a fair tontine, expected payout has a simple formula that depends only on a member's own parameters.

- An FTA is a fair tontine with self payback.

  - Noisy version of a fair annuity.

  - Outperforms an insurer-provided annuity.

# The Time Value of Money: Private Investment Accounts

In FTA, member contributions can be invested for growth over time.

- Each member manages her own portfolio of investments.
  - E.g., a brokerage account holding stocks, bonds, mutual fund, etc.
- For FTP, $s_i$'s are snapshots of portfolio values.
- A member's expected payout scales in proportion to his own portfolio's value.
  - Unaffected by other members' portfolios.

# A Big World of Providers

FTA could be offered by mutual fund houses, discount brokers, etc.

- No insurer needed.
- More investment choices.
  - E.g., member has a brokerage account to trade anything on the market.
  - Best-of-breed investment choices available.
- Arrangement resembling an IRA.

# It's Not Just Annuities

FTA easily modified to do other mortality-pooled products

- Deferred annuities, longevity insurance, fixed-term annuities, etc.
- All that changes is payout schedule
- Underlying fair tontine is unchanged
- Single tontine can support members with different products

# The End

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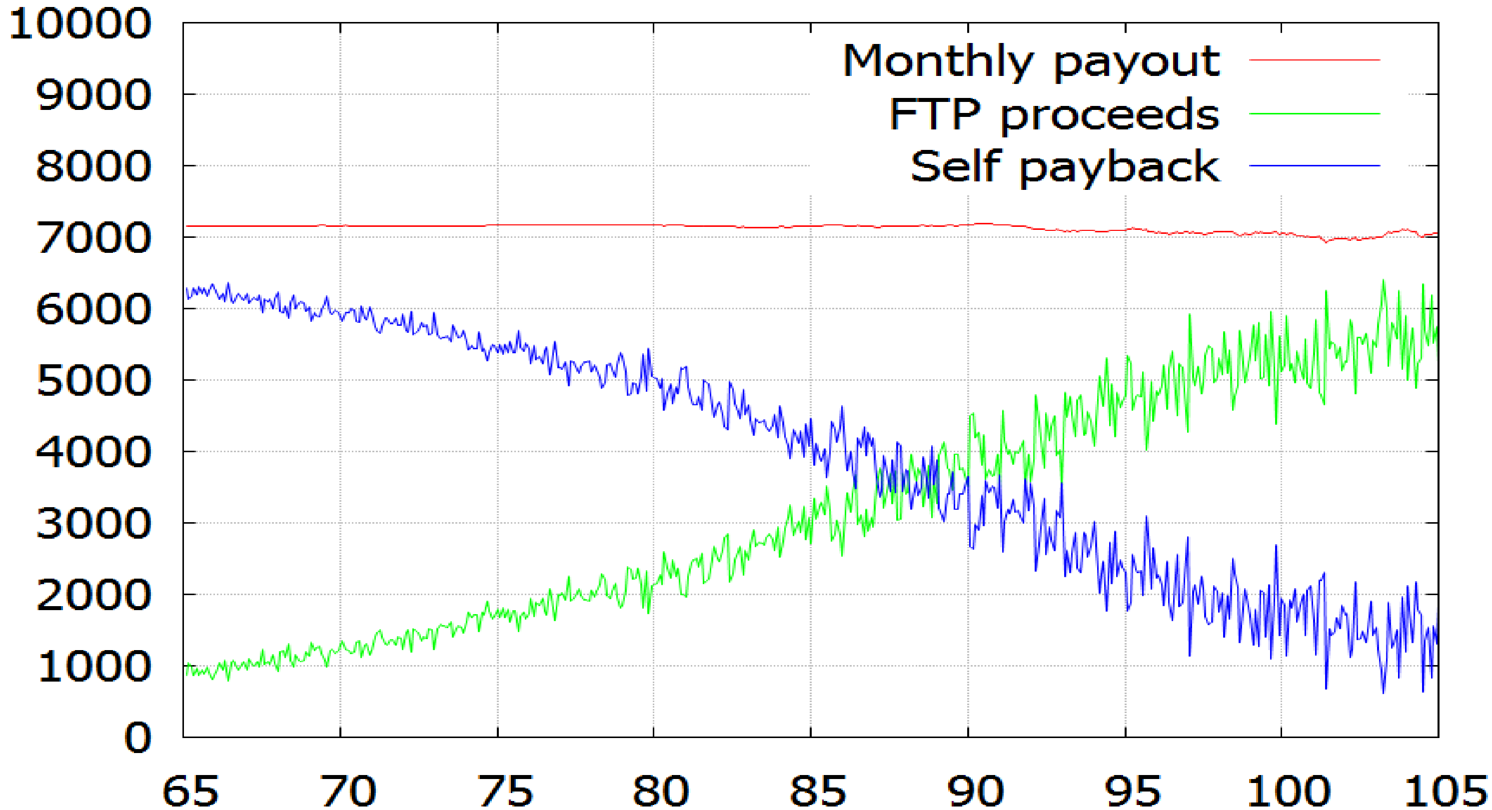
# Backup

# Historical Quote

"[I]t is very difficult to establish [tontines] on sound principles, or according to the rules deduced from the theory of probabilities. ... To establish a fair tontine, it would be indispensable to class together none but individuals of the same age. But it would be impossible to establish any extensive tontine upon such principles, that is, on principles that would render the chances of the subscribers equal, and fully worth the sum paid for them."

J.R. McCulloch. *A Treatise on the Principles and Practical Influence of Taxation and the Funding System.* Eyre and Spottiswoode, Her Majesty's Printers. Edinburgh, 1863.

# FTA Simulation, Smooth Payout Method




Payments received by a typical long-lived male.
About 75,000 members, wide range of ages, genders, and contributions.

# Bisection Algorithm to Build Separable FTP

Initialize $l = \theta_1$, $h = 2\theta_1$.
**do**

$w_1 \leftarrow (l + h)/2$

**for** $i = 2, \ldots, m$ **do**

$$w_i \leftarrow \frac{1}{2} - \frac{1}{2}\left[1 - \frac{4\theta_i}{\theta_1} w_1(1 - w_1)\right]^{1/2}$$

**done**

**if** $\sum_{i=1}^{m} w_i < 1$ **then** $l \leftarrow w_1$ **else** $h \leftarrow w_1$

**while** $(h - l)/l > \varepsilon$

Run time is $O(m \log(1/\varepsilon))$. For fixed level of precision, such as 64-bit floating point, run time is $O(m)$.

# Fair Tontine Theorem

**Theorem.** *In a fair tontine, a member with contribution* $s$ *who joins prior to* $t_1$ *and is alive at* $t_2$ *has an expected payout during* $(t_1, t_2)$ *of*

$$s \int_{t_1}^{t_2} \mu(t)\, dt.$$

*Proof.* Small interval $(t, t+\Delta)$ in $(t_1, t_2)$. $\Pr\{\tau < t+\Delta \,|\, \tau > t\} \approx \mu(t)\Delta$. Let $r(t)\Delta$ be expected payout for $(t, t+\Delta)$. By fairness,

$$-s\mu(t)\Delta + r(t)\Delta \underbrace{(1 - \mu(t)\Delta)}_{\approx 1} = 0,$$

so $r(t) \approx s\mu(t)$. Expected payout on $(t_1, t_2)$ is $\int_{t_1}^{t_2} r(t)\, dt = s \int_{t_1}^{t_2} \mu(t)\, dt$. □