Fair Tontine Annuity

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Agenda

- Background
  - **Annuities**
  - Tontines
  - Actuarial Preliminaries
- Fair Tontine
- Fair Tontine Annuity
- Concluding Remarks

What is the Problem?

- Individual retires with no pension, has to live off savings.

- Possible long life (50 years?), can't spend too fast.

- Likely shorter life (25 years?), lost opportunity to spend and enjoy.

- Need: a financial product to deal with uncertainty of lifetime.

What is an Annuity?

- Contract between a person (“annuitant”) and an insurer.
- At time 0, annuitant pays premium s to insurer.
- Thereafter, insurer makes monthly payment d to annuitant for life.
  - First payment at time T, where $T = 1/12$
  - Second payment at $2T$
  - Etc.
- Moral: Annuity is a purchased pension.

Expected Value of Monthly Payments

Expected value of payments is

$$\sum_{k=1}^{\infty} d\,\text{Pr}\{\text{alive at } kT \,|\, \text{alive at } 0\}$$

Annuity is *fair* if premium s equals expected value of payments.

Life Isn't Fair
and Neither Are Annuities

- In practice, insurer charges higher premium than fair.
  - Insurer expects to make a profit, not a loss.
  - A typical case: fair premium \$100,000, insurer charges \$115,000.
- Moral: Annuitant gets less in a practical annuity than in a fair annuity.
- Question: Can we make a *practical* fair annuity?

Remarks

- For simplicity, we ignore the time value of money.
  - I.e., we assume interest rate of 0%.

- The term "annuity" means many things in the insurance industry in addition to the definition here.
  - Caveat: If you talk with an insurance person about "annuities," she might be thinking something very different.

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What's a Tontine?

- Predecessor to modern annuities.
  - Invented in 1653 by Lorenzo de Tonti.
  - Popular in Europe and U.S. in 17th and 18th centuries.
- At time 0, each of m persons (members) contributes amount s.
- When a member dies, his s is divided among survivors.
  - After first death, each survivor receives $s/(m-1)$.
  - After second death, each survivor receives $s/(m-2)$.
  - Etc.
- Provides each member a lifetime payment stream.

What's Wrong with a Tontine?

- *Age discrimination.* Older members have lower expected payout.

- *Identical contribution level.* Too low for the rich, too high for the middle class.

- *Closed end.* All members join at time 0, then nobody else joins.

- *Random payment times.* Paid at times of member deaths.

- *Increasing payments.* Starve during early years.

What's Good About a Tontine? No Insurer

- No insurer profit.

- All money paid in by members gets distributed to members.

What We Are Going To Do

- Make the tontine fair for all ages and contributions.

- Wrap the tontine with features that make it resemble an annuity.

- Result: a Fair Tontine Annuity.
  - Insurer profit eliminated
  - "Noisy" version of a fair annuity
  - Suitable product for mutual fund houses, discount brokers, etc.

Historical Quote

"[I]t is very difficult to establish [tontines] on sound principles, or according to the rules deduced from the theory of probabilities. ... To establish a fair tontine, it would be indispensable to class together none but individuals of the same age. But it would be impossible to establish any extensive tontine upon such principles, that is, on principles that would render the chances of the subscribers equal, and fully worth the sum paid for them."

J.R. McCulloch. *A Treatise on the Principles and Practical Influence of Taxation and the Funding System.* Eyre and Spottiswoode, Her Majesty's Printers. Edinburgh, 1863.

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Mortality Table (excerpt)

Age	Male	Female	Age	Male	Female
	...			...	
55	0.004534	0.002457	56	0.004876	0.002689
57	0.005228	0.002942	58	0.005593	0.003218
59	0.005988	0.003523	60	0.006428	0.003863
	...			...	
85	0.073275	0.057913	86	0.080076	0.065119
87	0.087370	0.073136	88	0.095169	0.081991
89	0.103455	0.091577	90	0.112208	0.101758
	...			...	

- Entry for age n is

$$\Pr\{\text{die before birthday } (n+1) \,|\, \text{alive at birthday } n\}$$

- From Annuity 2000 mortality table.

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Tontine with Unequal Payouts

- Say we have m members, indexed $1, 2, \ldots, m$.
  (Different ages, genders, contributions.)

- Member i contributes s_i dollars.

- If member j dies, member i receives $\alpha_{ij} s_j$.

  - $0 \le \alpha_{ij} \le 1$ for $i \neq j$. Member i receives some fraction of j's contribution.

  - $\alpha_{jj} = -1$. Member j forfeits entire contribution.

  - $\sum_{i=1}^{m} \alpha_{ij} = 0$. Amounted forfeited by j equals amount received by surviving members.

Matrix Representation of α_{ij}'s

For example, $m = 5$:

$$
\begin{matrix}
-1 & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} & \alpha_{1,5} \\
\alpha_{2,1} & -1 & \alpha_{2,3} & \alpha_{2,4} & \alpha_{2,5} \\
\alpha_{3,1} & \alpha_{3,2} & -1 & \alpha_{3,4} & \alpha_{3,5} \\
\alpha_{4,1} & \alpha_{4,2} & \alpha_{4,3} & -1 & \alpha_{4,5} \\
\alpha_{5,1} & \alpha_{5,2} & \alpha_{5,3} & \alpha_{5,4} & -1
\end{matrix}
$$

- If member j dies, column j describes payouts.
  - Column elements sum to 0.

- Call the α_{ij}'s a *transfer plan*.

Making the Transfer Plan Fair

- Suppose at time t a member dies.
  - Pretend we don't know which member has died. Let

    $$p_j = \text{Pr}\{\text{member } j \text{ died at } t \,|\, \text{some member died at } t\}$$

  - Easy to compute p_j's from mortality table.

- Expected amount received by member i is

$$ER_i = \sum_{j=1}^{m} p_j \alpha_{ij} s_j.$$

- Plan is fair if $ER_i = 0$ for each member i.
  - We call any such plan a *Fair Transfer Plan (FTP)*.

Recap of FTP Conditions

$$\alpha_{jj} = -1 \text{ for } j = 1, 2, \ldots, m;$$
$$0 \le \alpha_{ij} \le 1 \text{ for } i, j = 1, 2, \ldots, m,\ i \neq j;$$
$$\sum_{i=1}^{m} \alpha_{ij} = 0 \text{ for } j = 1, 2, \ldots, m;$$
$$\sum_{j=1}^{m} \alpha_{ij} p_j s_j = 0 \text{ for } i = 1, 2, \ldots, m.$$

Does an FTP Exist?

- A necessary condition for FTP existence is:

$$0 = \sum_{j=1}^{m} \alpha_{ij} p_j s_j = -p_i s_i + \sum_{j \neq i} \alpha_{ij} p_j s_j \leq -p_i s_i + \sum_{j \neq i} p_j s_j,$$

  so

$$2 p_i s_i \leq \sum_{j=1}^{m} p_j s_j \text{ for } i = 1, 2, \ldots, m.$$

- The condition is also sufficient, but the proof is harder and omitted here.
  - Can be shown using network flow graph and max-flow min-cut theorem.

FTP Existence Theorem

Theorem. *An FTP exists if and only if*

$$p_i s_i \leq \frac{1}{2} \sum_{j=1}^{m} p_j s_j \textit{ for } i = 1, 2, \ldots, m.$$

- $p_i s_i$ is member i's risk of loss.
  - Elderly (high p_i) with large contribution (high s_i) at high risk of loss. Younger with small contribution at low risk.
- Theorem says risk of loss cannot be concentrated in any one member.
- Easy to meet in practice by capping contribution.

How to Construct an FTP

- Many solutions to FTP constraints, some better than others.

- A good solution: "separable" FTP.

  - Let $\alpha_{ij} = \dfrac{w_i}{(1 - w_j)}$ for $i \neq j$, each $w_i \geq 0$, $\sum_{i=1}^{m} w_i = 1$.

  - Such an FTP exists, is unique, and has good properties.

  - The w_i's can be computed in $O(m)$ run time.

How to Make a Fair Tontine

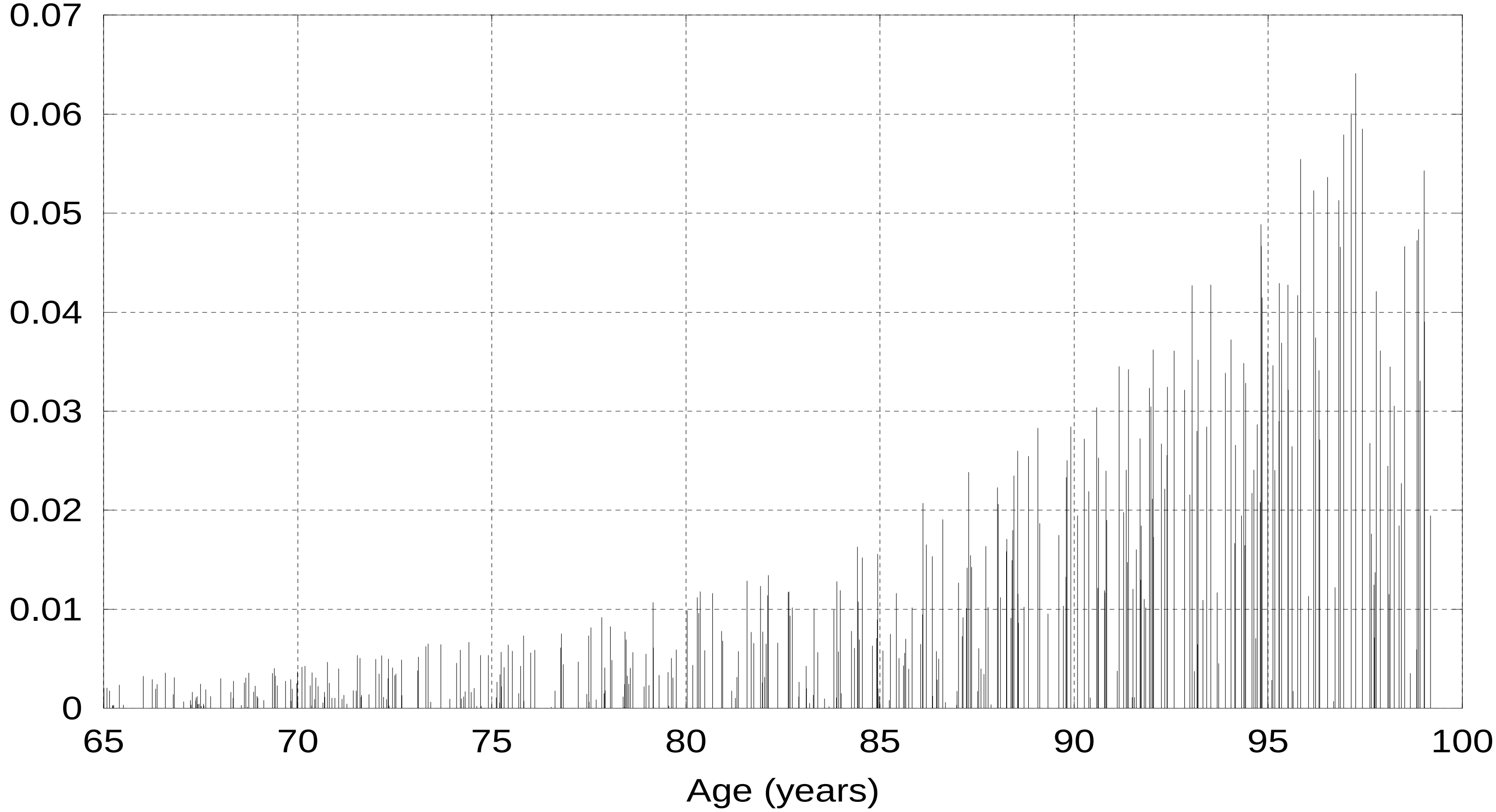
When a member dies:

- Count m, compute p_j's, construct an FTP.
- Distribute dying member's contribution according to FTP.

Accomplishes the following goals:

- *Age indiscriminate.* All ages treated fairly.
- *Individual contributions.* All contribution levels treated fairly.
- *Open ended.* New members may join at any time. Tontine operates in perpetuity.

Simulation Result



Payments received by a typical long-lived male, normalized to $1 contribution.
About 5000 members, wide range of ages, genders, and contributions.

What's Wrong with a Fair Tontine?
(Versus an Annuity)

- *Random payment times*
  (versus monthly).

- *Increasing payments*
  (versus uniform).

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How to Get Monthly Payments in a Tontine

- Each member has an account that holds her contribution.

- When a member dies, his account distributed using an FTP.
  - Distribution deposited into accounts of surviving members.
  - The s_i values for FTP are account balances, including any deposits from prior FTP's.

- At end of month, each living member is paid balance of her account in excess of her contribution.

Example Statement of Tontine Account: Living Member

Date	Amount	Balance	Description
03/31		250,000.00	
04/02	67.17	250,067.17	Proceeds from FTP
04/03	25.21	250,092.38	Proceeds from FTP
04/05	55.14	250,147.52	Proceeds from FTP
04/07	135.41	250,282.93	Proceeds from FTP
04/07	48.91	250,331.84	Proceeds from FTP
04/12	52.29	250,384.13	Proceeds from FTP
04/15	102.54	250,486.67	Proceeds from FTP
04/20	159.46	250,646.13	Proceeds from FTP
04/21	139.68	250,785.82	Proceeds from FTP
04/22	17.82	250,803.63	Proceeds from FTP
04/25	124.81	250,928.44	Proceeds from FTP
04/28	55.32	250,983.76	Proceeds from FTP
04/30	57.91	251,041.67	Proceeds from FTP
04/30	(1,041.67)	250,000.00	Payout of FTP proceeds

Example Statement of Tontine Account: Deceased Member

Date	Amount	Balance	Description
03/31		250,000.00	
04/02	67.17	250,067.17	Proceeds from FTP
04/03	25.21	250,092.38	Proceeds from FTP
04/05	55.14	250,147.52	Proceeds from FTP
04/07	135.41	250,282.93	Proceeds from FTP
04/07	48.91	250,331.84	Proceeds from FTP
04/12	(250,331.84)	0	Forfeited to FTP

Expected Value of Monthly Payment

Let:

$$q = \text{Pr}\{\text{die during month} \mid \text{alive at start of month}\}$$
$$r = E(\text{payment} \mid \text{alive at end of month})$$

If member dies during month, loses s;
if survives month, gains payment. By fairness:

$$-sq + r(1-q) = 0,$$

so

$$r = \frac{sq}{1-q}.$$

Surprise! Expected payment depends only on member's own age and gender (for q) and contribution s. Parameters of other members do not matter.

How to Get Uniform Expected Payments: “Self Payback”

- Each month, reduce a living member’s account balance by paying him a portion of initial contribution.
  - In addition to paying out FTP proceeds for the month.

- Combination of this (deterministic) self payback plus the (random) FTP proceeds has an expected value that is constant over member’s lifetime.

- This arrangement is called a *Fair Tontine Annuity* (FTA).

Example Statement of FTA Account: Living Member

Date	Amount	Balance	Description
03/31		250,000.00	
04/02	67.17	250,067.17	Proceeds from FTP
04/03	25.21	250,092.38	Proceeds from FTP
04/05	55.14	250,147.52	Proceeds from FTP
04/07	135.41	250,282.93	Proceeds from FTP
04/07	48.91	250,331.84	Proceeds from FTP
04/12	52.29	250,384.13	Proceeds from FTP
04/15	102.54	250,486.67	Proceeds from FTP
04/20	159.46	250,646.13	Proceeds from FTP
04/21	139.68	250,785.82	Proceeds from FTP
04/22	17.82	250,803.63	Proceeds from FTP
04/25	124.81	250,928.44	Proceeds from FTP
04/28	55.32	250,983.76	Proceeds from FTP
04/30	57.91	251,041.67	Proceeds from FTP
04/30	(1,041.67)	250,000.00	Payout of FTP proceeds
04/30	(452.18)	249,547.82	Self payback

How Much Self Payback in an FTA?

Define month n as $(nT, (n+1)T)$.
Let:

$$\begin{aligned} b_n &= \text{account balance at start of month } n; \\ D_n &= \text{payment at end of month } n; \\ d &= \text{desired value of } E(D_n \,|\, \text{alive at end of month } n). \end{aligned}$$

Self payback for month n is $b_n - b_{n+1}$.

Claim: correct amount of self payback occurs when

$$b_n = \sum_{k=n+1}^{\infty} d \,\Pr\{\text{alive at } kT \,|\, \text{alive at } nT\}.$$

How Much Self Payback, cont'd

Proof of claim. Let $p(k|n) = \text{Pr}\{\text{alive at } kT \,|\, \text{alive at } nT\}$. Then

$$b_n = \sum_{k=n+1}^{\infty} d\, p(k|n),$$

and

$$\begin{aligned} &E(D_n \,|\, \text{alive at end of month } n) \\ &\qquad = \underbrace{\frac{b_n(1 - p(n+1|n))}{p(n+1|n)}}_{\text{from FTPs}} + \underbrace{b_n - b_{n+1}}_{\text{self payback}} \\ &\qquad = \frac{b_n}{p(n+1|n)} - b_{n+1} \\ &\qquad = \sum_{k=n+1}^{\infty} d\, p(k|n+1) - \sum_{k=n+2}^{\infty} d\, p(k|n+1) \\ &\qquad = d. \end{aligned}$$

FTA Theorem

Notice that the starting balance is:

$$b_0 = \sum_{k=1}^{\infty} d \, \mathsf{Pr}\{\text{alive at } kT \,|\, \text{alive at } 0\}.$$

Surprise! b_0 is identical to premium of a fair annuity having monthly payment d.

Thus we have established:

Theorem. *An FTA with initial contribution s has a random monthly payment whose expected value is identical to the fixed monthly payment of a fair annuity purchased for premium s.*

Remarks on FTA Theorem

- FTA is a noisy version of a fair annuity:
  expected monthly payment of FTA
  $=$ monthly payment of fair annuity.

- Expected monthly payment of FTA
  $>$ monthly payment of insurer-provided annuity

- Law of large numbers, for large pool of members:
  monthly payment of FTA
  $\approx$ monthly payment of fair annuity.

Simulation

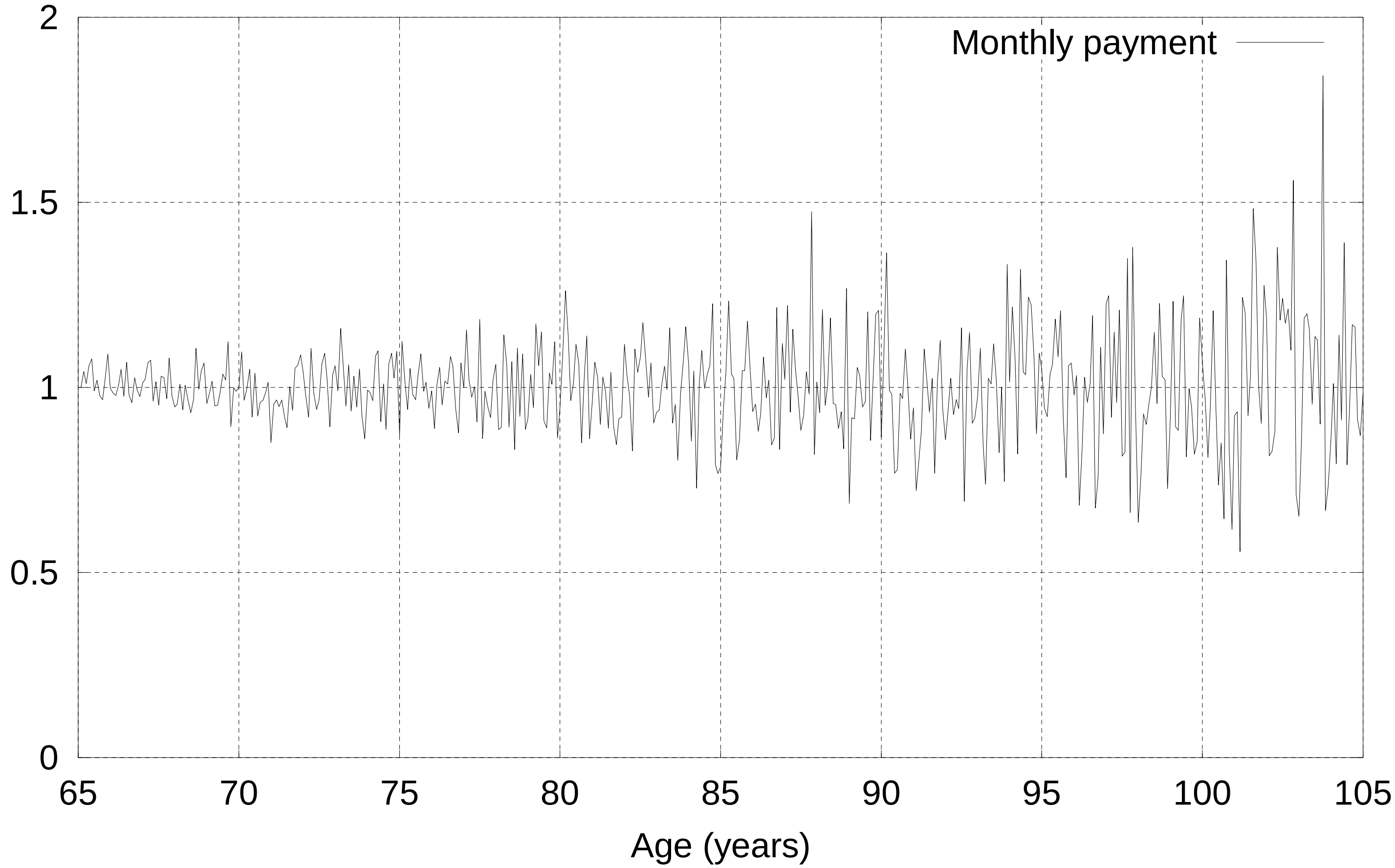
- 50 year simulation

- Twenty members join each month, each age 65

  - Random gender, random contribution in range \$100 to \$100,000

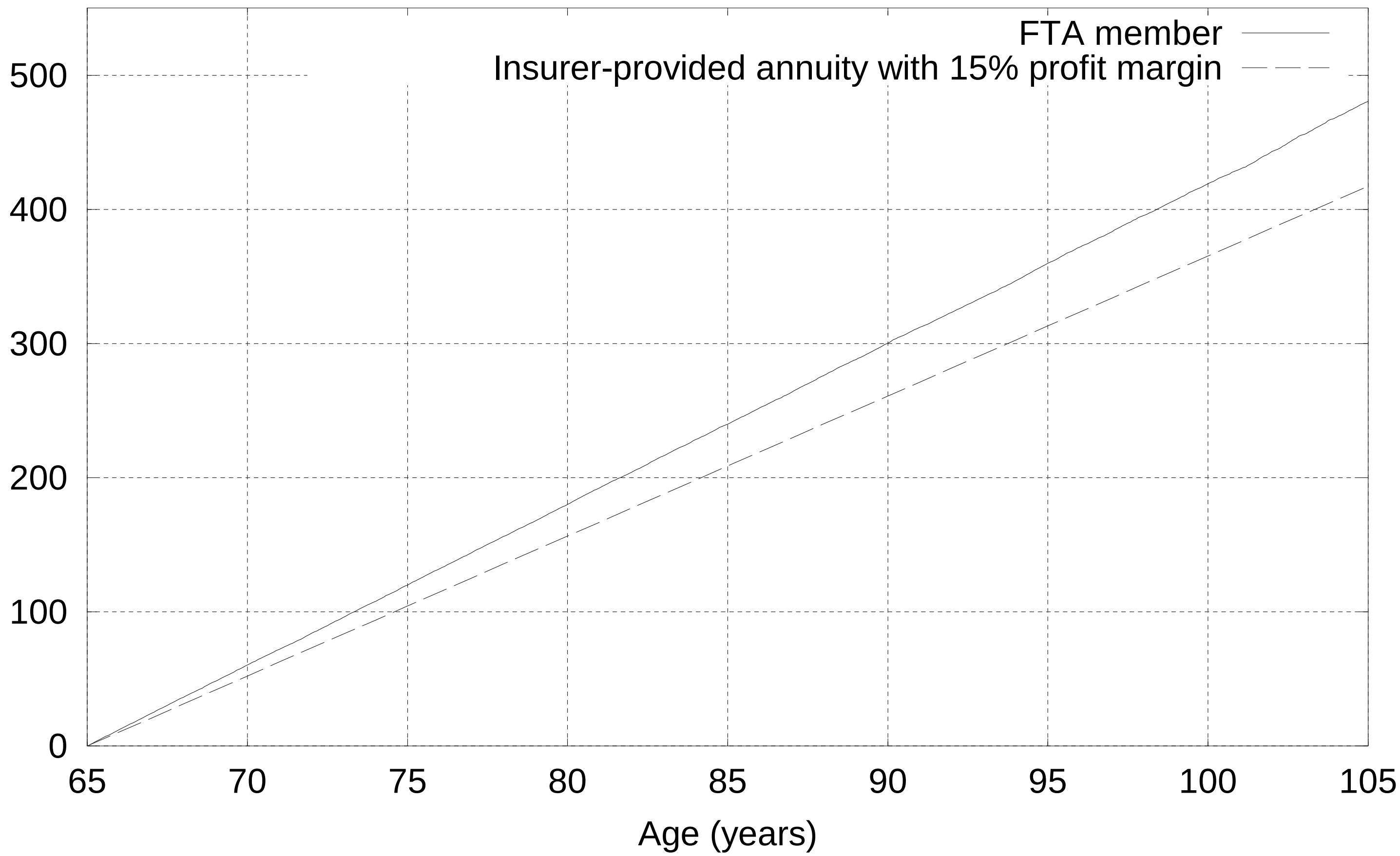
  - Average number of living members about 5,200.

Simulation Result



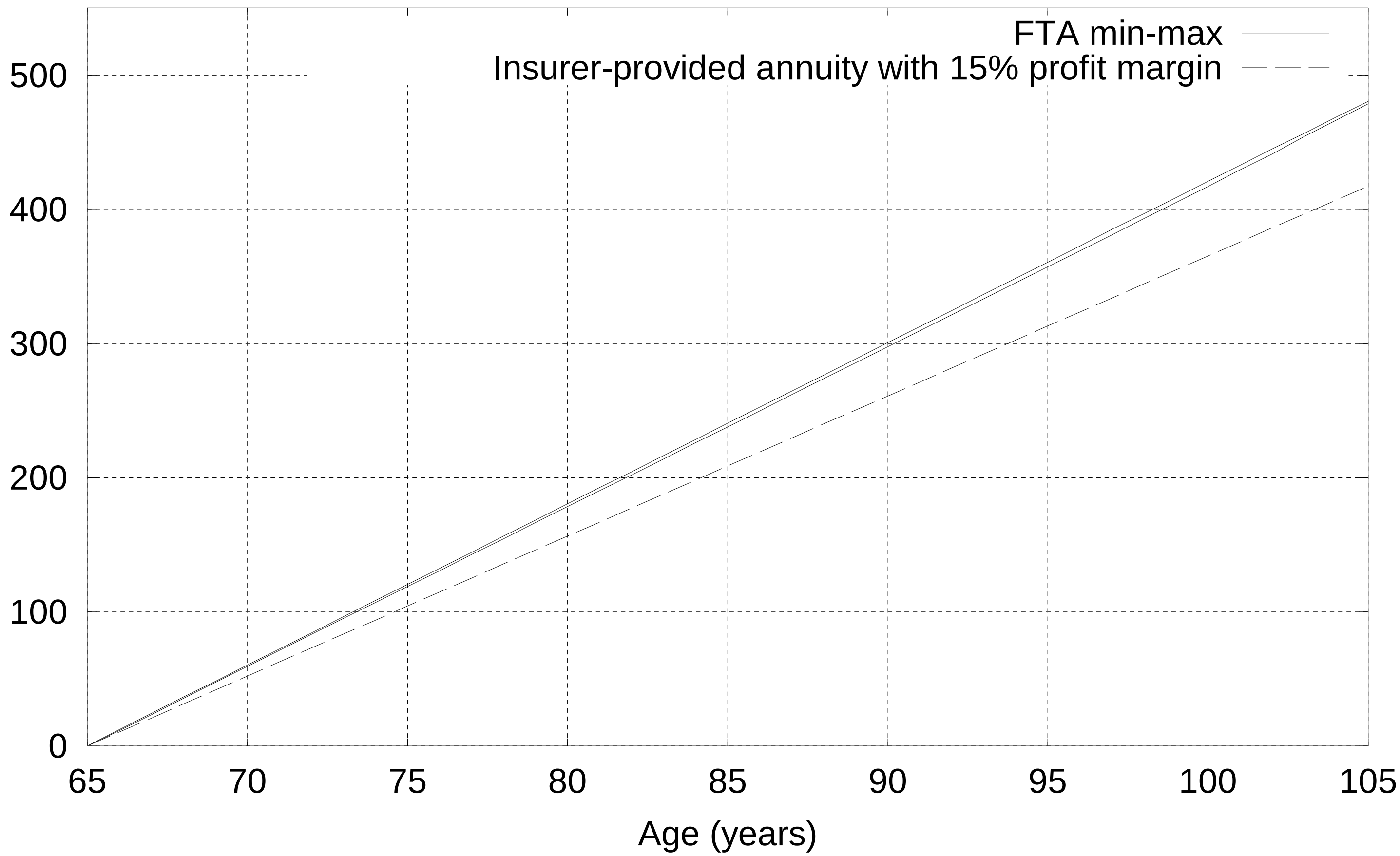
Monthly payment for a typical long-lived member, normalized to $1 expected value, versus age.

Simulation Result



Accumulated normalized payout versus age: a typical long-lived FTA member; insurer annuity with typical profit margin.

Simulation Result



Accumulated normalized payout versus age: max-min over all FTA members; insurer annuity with typical profit margin.

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Recap

- A tontine can be made fair using FTPs.
- FTP exists if no one member's assets dominate.
  - Easily constructed (e.g., separable algorithm).
- In a fair tontine, expected payout has a simple formula that depends only on a member's own parameters.
- An FTA is a fair tontine with self payback.
  - Noisy version of a fair annuity.
  - Outperforms an insurer-provided annuity.

The Time Value of Money: Private Investment Accounts

In FTA, member contributions can be invested for growth over time.

- Each member manages her own portfolio of investments.
  - E.g., a brokerage account holding stocks, bonds, mutual fund, etc.

- For FTP, s_i's are snapshots of portfolio values.

- A member's expected payout scales in proportion to his own portfolio's value.
  - Unaffected by other members' portfolios.

A Big World of Providers

FTA could be offered by mutual fund houses, discount brokers, etc.

- No insurer needed.
- More investment choices.
  - E.g., member has a brokerage account to trade anything on the market.
  - Best-of-breed investment choices available.
- Arrangement resembling an IRA.

It's Not Just Annuities

FTA easily modified to do other mortality-pooled products

- Deferred annuities, longevity insurance, fixed-term annuities, etc.
- All that changes is payout schedule
- Underlying fair tontine is unchanged
- Single tontine can support members with different products

The End

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Backup

Expected Value of Monthly Payments

Let τ be random variable representing time of death with distribution function $F(t) = \text{Pr}\{\tau \leq t\}$.

Expected value of payments is

$$\sum_{k=1}^{\infty} d \cdot \text{Pr}\{\tau > t_0 + kT \,|\, \tau > t_0\} =$$
$$d \underbrace{\sum_{k=1}^{\infty} \frac{1 - F(t_0 + kT)}{1 - F(t_0)}}_{L(t_0)} .$$

Annuity is *fair* if $s = dL(t_0)$.

Risk of Insurer Default

- Monthly payments are an obligation of insurer.

- If insurer defaults, annuitant might not receive all monthly payments.

- Question: Can we make a practical annuity that has no risk of insurer default?

Mortality Table (excerpt)

Age	Male	Female	Age	Male	Female
	...			...	
55	0.004534	0.002457	56	0.004876	0.002689
57	0.005228	0.002942	58	0.005593	0.003218
59	0.005988	0.003523	60	0.006428	0.003863
	...			...	
85	0.073275	0.057913	86	0.080076	0.065119
87	0.087370	0.073136	88	0.095169	0.081991
89	0.103455	0.091577	90	0.112208	0.101758
	...			...	

- Each entry is $\Pr\{n < \tau \leq n+1 \mid \tau > n\}$.

- From Annuity 2000 mortality table.

Constructing $F(t)$ from Mortality Table

- Each entry in table is

$$\Pr\{n < \tau \leq n + 1 \,|\, \tau > n\} = \frac{F(n+1) - F(n)}{1 - F(n)}.$$

- Start with $F(0) = 0$. Recursively construct $F(n)$ for $n = 1, 2, \ldots$, using table entries.

- Interpolate between $F(n)$ and $F(n+1)$ for $n < t < n + 1$.

Interpolating $F(t)$: Constant Force of Death

- Let $f(t) = F'(t)$ (density function).

- Let $\mu(t) = \dfrac{f(t)}{1 - F(t)}$ ("force of death" function).
  For small Δ,

$$\Pr\{t < \tau < t + \Delta \,|\, \tau > t\} \approx \Delta\mu(t).$$

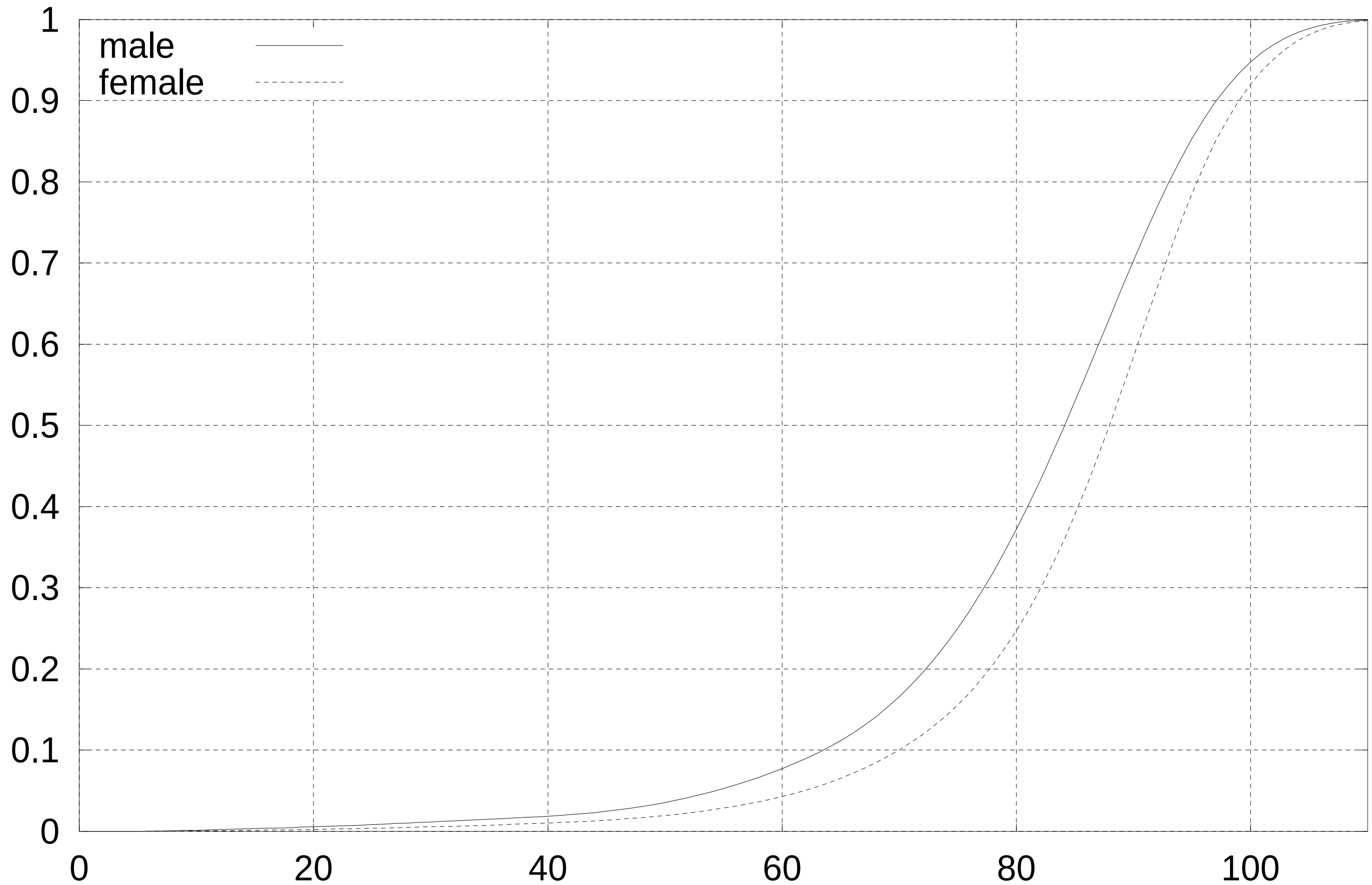
- Define interpolation such that $\mu(t)$ is constant for $n < t < n + 1$. Since

$$\mu(t) = -\frac{d}{dt}\ln(1 - F(t)),$$

  then

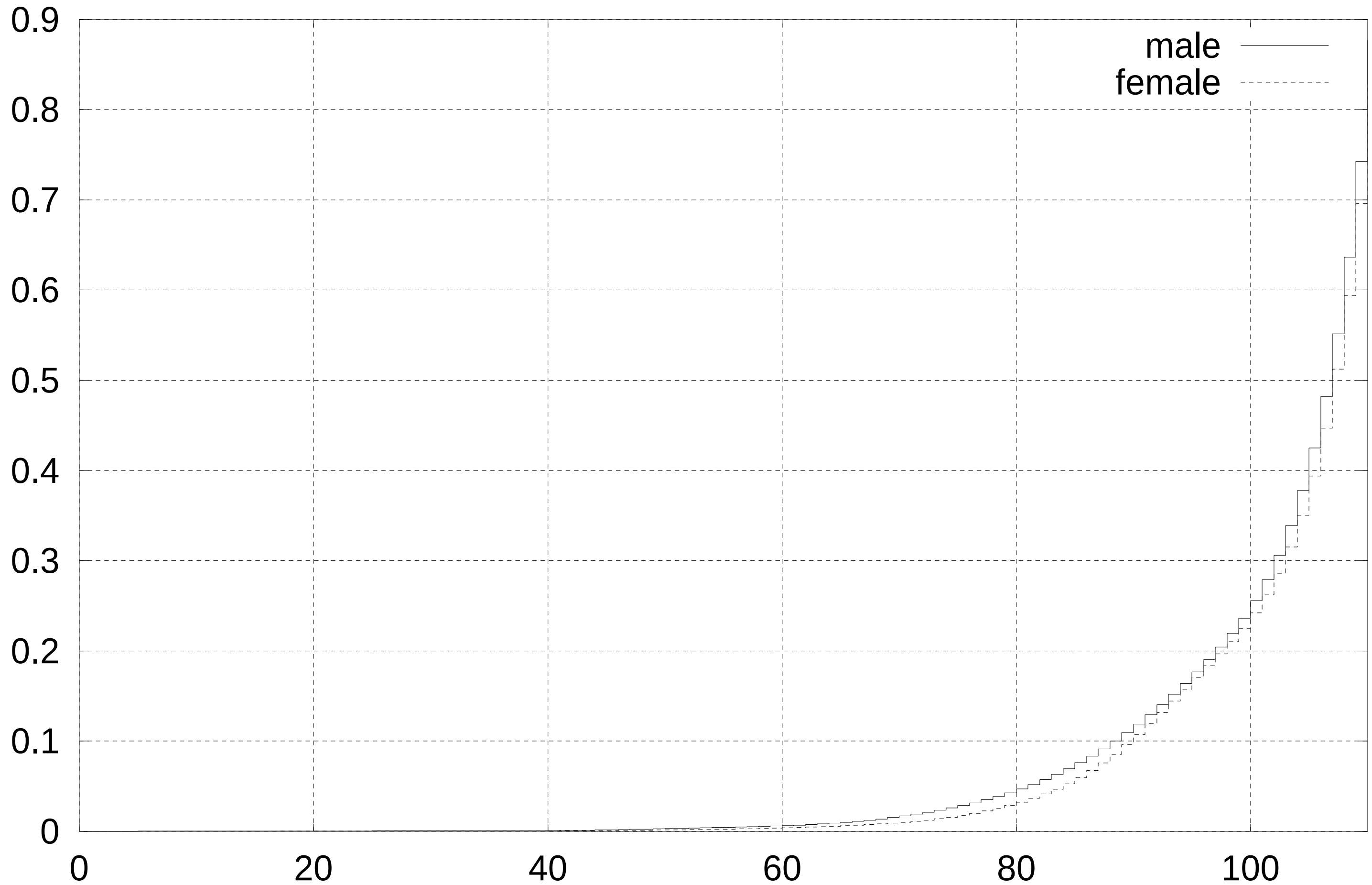
$$1 - F(t) = (1 - F(n))\left[\frac{1 - F(n+1)}{1 - F(n)}\right]^{t-n}.$$

Plot of $F(t)$



Horizontal axis is age t in years, vertical axis is $F(t)$.

Plot of $\mu(t)$



Horizontal axis is age t in years, vertical axis is $\mu(t)$.

Let's Play a Game

- Say we have m members, indexed $1, 2, \ldots, m$.
  (Different ages, genders.)

- Member i contributes s_i dollars.

- Suppose at time t a member dies.

  - Pretend we don't know which member has died.
    Let p_j be (conditional) probability that j has died.

$$p_j = \frac{u_j(t)}{\sum_{i=1}^{m} u_i(t)}.$$

- Is there a fair way to distribute dying member's contribution?

Matrix Representation of FTP

Example: $m = 5$:

$$\begin{matrix} -1 & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} & \alpha_{1,5} \\ \alpha_{2,1} & -1 & \alpha_{2,3} & \alpha_{2,4} & \alpha_{2,5} \\ \alpha_{3,1} & \alpha_{3,2} & -1 & \alpha_{3,4} & \alpha_{3,5} \\ \alpha_{4,1} & \alpha_{4,2} & \alpha_{4,3} & -1 & \alpha_{4,5} \\ \alpha_{5,1} & \alpha_{5,2} & \alpha_{5,3} & \alpha_{5,4} & -1 \end{matrix}$$

Proof of FTP Existence Theorem, “If”

- Harder than “only if.”

- A nice argument can be made by formulating as a network flow problem and applying max-flow min-cut theorem.

- Instead, here we will prove “if” by constructing a specific FTP, which we want anyway.

Proof of "If", Preliminaries

Let $\theta_i = p_i s_i$. Without loss of generality, assume:

$$\theta_1 \geq \theta_2 \geq \cdots \geq \theta_m;$$

$$\sum_{i=1}^{m} \theta_i = 1.$$

Will show that FTP exists if

$$\theta_1 \leq \frac{1}{2}.$$

Proof of "If," Extreme Case

Suppose $\theta_1 = 1/2$. Then an FTP (for example $m = 5$) is

$$\begin{array}{rrrrr} -1 & 1 & 1 & 1 & 1 \\ 2\theta_2 & -1 & 0 & 0 & 0 \\ 2\theta_3 & 0 & -1 & 0 & 0 \\ 2\theta_4 & 0 & 0 & -1 & 0 \\ 2\theta_5 & 0 & 0 & 0 & -1 \end{array}$$

$j = 1$: $\sum_i \alpha_{ij} = -1 + 2\sum_{i=2}^m \theta_i = -1 + 2(1 - \theta_1) = 0.$

$j > 1$: $\sum_i \alpha_{ij} = 1 - 1 = 0.$

$i = 1$: $\sum_j \alpha_{ij}\theta_j = -\theta_1 + \sum_{j=2}^m \theta_j = -\theta_1 + (1 - \theta_1) = 0.$

$i > 1$: $\sum_j \alpha_{ij}\theta_j = 2\theta_i\theta_1 - \theta_i = 0.$

Remark: this is the only FTP for $\theta_1 = 1/2$.

Proof of "If," Nonextreme Case

A *separable* FTP has the following form:

- Assign a non-negative weight w_i to each member i, $0 \leq w_i < 1$, $\sum_{i=1}^{m} w_i = 1$.

- Define $\alpha_{ij} = \dfrac{w_i}{1 - w_j}$ for $i \neq j$. Like this (example $m = 4$):

$$\begin{matrix} -1 & \frac{w_1}{1-w_2} & \frac{w_1}{1-w_3} & \frac{w_1}{1-w_4} \\ \frac{w_2}{1-w_1} & -1 & \frac{w_2}{1-w_3} & \frac{w_2}{1-w_4} \\ \frac{w_3}{1-w_1} & \frac{w_3}{1-w_2} & -1 & \frac{w_3}{1-w_4} \\ \frac{w_4}{1-w_1} & \frac{w_4}{1-w_2} & \frac{w_4}{1-w_3} & -1 \end{matrix}$$

Will show a separable FTP exists if $\theta_1 < 1/2$.

Separable FTP, Column Sum

$$\begin{matrix} -1 & \frac{w_1}{1-w_2} & \frac{w_1}{1-w_3} & \frac{w_1}{1-w_4} \\ \frac{w_2}{1-w_1} & -1 & \frac{w_2}{1-w_3} & \frac{w_2}{1-w_4} \\ \frac{w_3}{1-w_1} & \frac{w_3}{1-w_2} & -1 & \frac{w_3}{1-w_4} \\ \frac{w_4}{1-w_1} & \frac{w_4}{1-w_2} & \frac{w_4}{1-w_3} & -1 \end{matrix}$$

Column j:

$$\sum_{i=1}^{m} \alpha_{ij} = -1 + \sum_{i \neq j} \frac{w_i}{1 - w_j} = -1 + \frac{1}{1 - w_j} \underbrace{\sum_{i \neq j} w_i}_{1-w_j} = 0,$$

as required for FTP.

Separable FTP, Row Sum

$$\begin{matrix} -1 & \frac{w_1}{1-w_2} & \frac{w_1}{1-w_3} & \frac{w_1}{1-w_4} \\ \frac{w_2}{1-w_1} & -1 & \frac{w_2}{1-w_3} & \frac{w_2}{1-w_4} \\ \frac{w_3}{1-w_1} & \frac{w_3}{1-w_2} & -1 & \frac{w_3}{1-w_4} \\ \frac{w_4}{1-w_1} & \frac{w_4}{1-w_2} & \frac{w_4}{1-w_3} & -1 \end{matrix}$$

Trick: if w_i's satisfy

$$\theta_i w_j(1 - w_j) = \theta_j w_i(1 - w_i) \text{ for } i \neq j,$$

then row sum i is:

$$-\theta_i + \sum_{j \neq i} \frac{w_i \theta_j}{1 - w_j} = -\theta_i + \sum_{j \neq i} \frac{w_j \theta_i}{1 - w_i} = -\theta_i + \frac{\theta_i}{1 - w_i} \underbrace{\sum_{j \neq i} w_j}_{1-w_i} = 0.$$

Separable FTP, Row Sum cont'd

Solve for w_i, $\theta_1 w_i(1-w_i) = \theta_i w_1(1-w_1)$:

$$w_i = \frac{1}{2} - \frac{1}{2}\left[1 - \frac{4\theta_i}{\theta_1} w_1(1-w_1)\right]^{1/2}, \; i > 1.$$

Then $\sum_{i=1}^{m} w_i = g(w_1)$, where

$$g(x) = x + \sum_{i=2}^{m} \left(\frac{1}{2} - \frac{1}{2}\left[1 - \frac{4\theta_i}{\theta_1} x(1-x)\right]^{1/2} \right).$$

Separable FTP exists if $g(w_1) = 1$ for some $w_1 \in [0,1)$.
Note that $g(0) = 0$ and $g(1) = 1$. After some effort,

$$g'(1) = 1 - \frac{1}{\theta_1} \underbrace{\sum_{i=2}^{m} \theta_i}_{1-\theta_1} = 2 - \frac{1}{\theta_1} < 0$$

for $\theta_1 < 1/2$. Thus $g(x) > 1$ for some $x \in (0,1)$. Since $g(0) = 0$ and g is continuous, $g(w_1) = 1$ for some $w_1 \in (0,1)$.

Bisection Algorithm to Build Separable FTP

Initialize $l = \theta_1$, $h = 2\theta_1$.

do

$w_1 \leftarrow (l + h)/2$

for $i = 2, \ldots, m$ **do**

$$w_i \leftarrow \frac{1}{2} - \frac{1}{2}\left[1 - \frac{4\theta_i}{\theta_1} w_1(1 - w_1)\right]^{1/2}$$

done

if $\sum_{i=1}^{m} w_i < 1$ **then** $l \leftarrow w_1$ **else** $h \leftarrow w_1$

while $(h - l)/l > \varepsilon$

Run time is $O(m \log(1/\varepsilon))$. For fixed level of precision, such as 64-bit floating point, run time is $O(m)$.

Separable FTP, Recap

- Separable FTP exists if $\theta_1 < 1/2$.
  - Completes proof of FTP Existence Theorem.

- Separable FTP is good practical choice for tontine problem.
  - Each surviving member i receives in proportion to fixed weight w_i.
  - Easy to construct, $O(m)$ run time.

Fair Tontine Theorem

Theorem. *In a fair tontine, a member with contribution* s *who joins prior to* t_1 *and is alive at* t_2 *has an expected payout during* (t_1, t_2) *of*

$$s \int_{t_1}^{t_2} \mu(t)\, dt.$$

Proof. Small interval $(t, t+\Delta)$ in (t_1, t_2). $\Pr\{\tau < t+\Delta \,|\, \tau > t\} \approx \mu(t)\Delta$. Let $r(t)\Delta$ be expected payout for $(t, t+\Delta)$. By fairness,

$$-s\mu(t)\Delta + r(t)\Delta \underbrace{(1-\mu(t)\Delta)}_{\approx 1} = 0,$$

so $r(t) \approx s\mu(t)$. Expected payout on (t_1, t_2) is $\int_{t_1}^{t_2} r(t)\, dt = s\int_{t_1}^{t_2} \mu(t)\, dt$. $\square$

Fair Tontine Theorem

Theorem. *In a fair tontine, a member with contribution s who joins prior to t_1 and is alive at t_2 has an expected payout during (t_1, t_2) of*

$$-s \ln\left(\Pr\{\textit{alive at } t_2 \,|\, \textit{alive at } t_1\}\right).$$

- Observation: member's expected payout depends only on his/her own parameters (age, gender, contribution). It does not depend on the parameters of other members.
  - Expected loss from dying depends only on member's own parameters.
  - By fairness, expected payout for surviving matches expected loss from dying.

Remarks on Fair Tontine Theorem

- Expected payout on (t_1, t_2) is

$$s \int_{t_1}^{t_2} \frac{f(t)}{1 - F(t)}\, dt = s \ln \left(\frac{1 - F(t_1)}{1 - F(t_2)} \right)$$
$$= -s \ln \Pr\{\tau > t_2 \,|\, \tau > t_1\}.$$

- A member's expected payout depends only on his/her own parameters (age, gender, contribution). It does not depend on the parameters of other members.
  - Expected loss from dying depends only on member's own parameters.
  - By fairness, expected payout for surviving matches expected loss from dying.

How Much Self Payback in an FTA?

- Recall that fair premium for \$1/month annuity purchased at time $t_0 + nT$ is

$$L(t_0 + nT) = \sum_{k=n+1}^{\infty} \frac{1 - F(t_0 + kT)}{1 - F(t_0 + nT)}.$$

  Remark that this decreases with n.

- The self payback at end of month n is defined as the amount that reduces the balance to

$$s \frac{L(t_0 + nT)}{L(t_0)},$$

  where s is original contribution.